SEMILINEAR EQUATIONS, THE γ_k FUNCTION, AND GENERALIZED GAUDUCHON METRICS

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ABSTRACT. In this paper, we generalize the Gauduchon metrics on a compact complex manifold and define the γ_k functions on the space of its hermitian metrics.

1. Introduction

Let X be a compact n-dimensional complex manifold. Let g be a hermitian metric on X and ω its hermitian form. It is well known that if $d\omega = 0$, then g or ω is called a Kähler metric and therefore X is called a Kähler manifold. When X is a non-Kähler manifold, one can consider the other conditions on ω such as

$$d\omega^k = 0, \qquad 2 \le k \le n-1. \tag{1.1}$$

If $d(\omega^{n-1}) = 0$, then g or ω is called a balanced metric and so X is called a balanced manifold [19]. However, when $2 \le k \le n-2$, $d\omega^k = 0$ automatically yields $d\omega = 0$ [15]. Instead of (1.1), one can consider the k-Kähler condition [1]. A complex manifold is called k-Kähler if it admits a closed complex transverse (k, k)–form. By this definition, a complex manifold is 1-Kähler if and only if it is Kähler; it is $(n-1)$–Kähler if and only if it is balanced.

One can also generalize the Kähler condition along other directions, for instance,

$$\partial\bar{\partial}\omega^k = 0, \qquad 1 \le k \le n-1. \tag{1.2}$$

When $k = n-1$, the metric ω is called a *Gauduchon* metric. Gauduchon [11] proved an interesting result that, for any hermitian metric ω on a compact complex n-dimensional manifold X, there exists a unique (up to a constant) smooth function v such that

$$\partial\bar{\partial}(e^v\omega^{n-1}) = 0 \qquad \text{on } X. \tag{1.3}$$

Thus, the Gauduchon metric always exists on a compact complex manifold. It is important in complex geometry since one can use such a metric to define the degree, and then make sense of the stability of holomorphic vector bundles over a non-Kähler complex manifold (see [18]).

When $k = n-2$, the metric ω satisfying (1.2) is called an *astheno-Kähler* metric. Jost and Yau [17] used this condition to study hermitian harmonic maps, and extended Siu's rigidity theorem to non-Kähler complex manifolds.

When $k = 1$, the metric ω in (1.2) is called a *pluriclosed* metric, which is also called strong KT (Kähler with torsion) metric (see [13, 7] and the references therein). Such a condition appeared in [6, 2] as a technical condition. Recently, Streets and

Tian [21] introduced a hermitian Ricci flow under which the pluriclosed metric is preserved.

It is important to find specific hermitian metrics on non-Kähler complex manifolds. J. Li, S.-T. Yau and Fu [8] have constructed balanced metrics on complex structures of manifolds $\#_{k\geq 2}(S^3\times S^3)$ which are obtained from the conifold transition of Calabi-Yau threefolds. As a corollary, there exists no pluriclosed metric on such manifolds. We note here that the specific hermitian geometry of threefolds $\#_k(\mathbb{S}^3\times\mathbb{S}^3)$ was first considered by Bozhkov [3, 4]. In this paper, we generalize (1.2) to weaker conditions:

$$\partial\bar\partial\omega^k\wedge\omega^{n-k-1}=0,\quad 1\leq k\leq n-1. \tag{1.4}$$

Definition 1. *Let ω be a hermitian metric on an n-dimensional complex manifold X, and k be an integer such that $1\leq k\leq n-1$. We call ω the k-th Gauduchon metric if ω satisfies (1.4).*

Note that an $(n-1)$-th Gauduchon metric is the classic Gauduchon metric. The natural question is whether there exists any k-th Gauduchon metric, $1\leq k\leq n-2$, on a complex manifold. To answer this question, one way is to look for such a metric in the conformal class of a given hermitian metric ω on X:

$$\partial\bar\partial(e^v\omega^k)\wedge\omega^{n-k-1}=0. \tag{1.5}$$

However, equation (1.5) in general needs not admit a solution (see below for reasons). In this paper, we solve the equation

$$\partial\bar\partial(e^v\omega^k)\wedge\omega^{n-k-1}=\gamma_k e^v\omega^n \tag{1.6}$$

for some constant γ_k satisfying the compatibility condition. The constant γ_k, if nonzero, can be viewed as an obstruction for the existence of a k-th Gauduchon metric in the conformal class of ω, for $1\leq k<n-1$.

Equation (1.6) can be reformulated, in a slightly more general form, as follows: Let (X,ω) be an n-dimensional compact hermitian manifold, and B be a smooth real 1-form on X. For any smooth function f on X satisfying

$$\int_X f\omega^n=0, \tag{1.7}$$

we consider the following semilinear equation

$$\Delta v+|\nabla v|^2+\langle B,dv\rangle=f\qquad\text{on } X. \tag{1.8}$$

Here Δ and ∇ are, respectively, the Laplacian and covariant differentiation associated with ω. Clearly, equation (1.8) needs not have a solution, due to the compatibility condition (1.7). For instance, let ω be balanced and $B=0$, then in order that (1.8) has a solution the function f has to be zero. Nonetheless, we shall show that, there is a smooth function v so that equation (1.8) holds up to a unique constant c. More generally, we have the following result:

Theorem 2. *Let (X,ω) be a compact hermitian manifold, B be a smooth real 1-form on X, and $\psi\in C^\infty(\mathbb{R})$ satisfy*

$$\liminf_{t\to+\infty}\frac{\psi(t)}{t^\mu}\geq\nu>0,\quad\text{where } \mu>1/2 \text{ and } \nu \text{ are constants.} \tag{1.9}$$

Then, for each $f \in C^\infty(X)$ satisfying (1.7)*, there exists a unique constant c, and a smooth function v on X, unique up to a constant, such that*

$$\Delta v + \psi(|\nabla v|^2) + \langle B, dv\rangle = f + c \qquad \text{on } X. \tag{1.10}$$

Remark 3. *The compatibility condition of* (1.10) *implies that*

$$c = \frac{\int_X (\Delta v + \psi(|\nabla v|^2) + \langle B, dv\rangle)\,\omega^n}{\int_X \omega^n},$$

which in general is nonzero.

Letting $\psi(t) = t$ on $\mathbb{R}$, we obtain an application of Theorem 2:

Corollary 4. *Let (X, ω) be an n-dimensional compact hermitian manifold. For any integer $1 \le k \le n-1$, there exists a unique constant γ_k, and a function $v \in C^\infty(X)$ satisfying that*

$$(\sqrt{-1}/2)\partial\bar\partial(e^v\omega^k) \wedge \omega^{n-k-1} = \gamma_k e^v \omega^n. \tag{1.11}$$

The solution v of (1.11) *is unique up to a constant. In particular, when $k = n-1$ we have $\gamma_{n-1} = 0$. If ω is Kähler, then $\gamma_k = 0$ and v is a constant, for each $1 \le k \le n-1$.*

Remark 5. *When $k = n-1$, this corollary recovers the classical result of Gauduchon* [11]*.*

By Corollary 4, we can associate each hermitian metric ω a unique constant $\gamma_k(\omega)$. Clearly, $\gamma_k = \gamma_k(\omega)$ is invariant under biholomorphisms. Furthermore, we will prove that γ_k depends smoothly on the hermitian metric ω (see Proposition 9); and that $\gamma_k(\omega) = 0$ if and only if there exists a k-th Gauduchon metric in the conformal class of ω (Proposition 8).

We will prove in Proposition 11 that the sign of $\gamma_k(\omega)$, denoted by $(\mathrm{sgn}\gamma_k)(\omega)$, is invariant in the conformal class of ω. We denote by $\Xi_k(X)$ the range of $\mathrm{sgn}\gamma_k$. By definition $\Xi_k(X) \subset \{-1, 0, 1\}$ for each k, and by Corollary 4 we have $\Xi_{n-1}(X) = \{0\}$. A natural question is *whether $\Xi_k(X) = \{-1, 0, 1\}$ for any $1 \le k \le n-2$ on any compact complex manifold X*. Indeed, if $\Xi_k(X) \supset \{-1, 1\}$ then the answer is positive, by Proposition 9. Thus, there will be a k-th Gauduchon metric on X. We can also ask whether $\Xi_k(X)$ is invariant under the modification. These questions will be systematically studied later. As a first step, we obtain the following result.

Theorem 6. *For $n = 3$, we have $1 \in \Xi_1(X)$. Namely, for any 3-dimensional hermitian manifold X, there exists a hermitian metric ω such that $\gamma_1(\omega) > 0$. In particular, there is no 1-st Gauduchon metric in the conformal class of ω.*

Then, we combine the above results to prove that, as an example, $\Xi_1 = \{-1, 0, 1\}$ on the three-dimensional complex manifolds constructed by Calabi [5]. As a consequence, there exists a 1-st Gauduchon metric on these manifolds. It is well-known that such manifolds are non-Kähler but admit balanced metrics. We do not know whether there exists any pluriclosed metric on them.

Another example we considered is $Y = S^5 \times S^1$, endowed with a complex structure so that the natural projection $\pi : S^5 \times S^1 \to \mathbb{P}^2$ is holomorphic. This would imply

that there is no balanced metrics on $S^5 \times S^1$. Moreover, we can prove that $S^5 \times S^1$ does not admit any pluriclosed metric. On the other hand, by considering a natural hermitian metric on $S^5 \times S^1$, we are able to show that $\Xi_1(S^5 \times S^1) = \{-1, 0, 1\}$. Thus, $S^5 \times S^1$ admits a 1-st Gauduchon metric.

We shall solve equation (1.10) by the continuity method. In Section 2, we set up the machinery and prove the openness. The closedness and *a priori* estimates are established in Section 3. In Section 4, we prove the uniqueness part in Theorem 2 and also prove Corollary 4. In Section 5, we discuss the relation between γ_k and the k-th Gauduchon metric. In section 6, we prove Theorem 6, and explicitly construct a metric with positive γ_1 on the complex 3-torus. As another example, we show that the natural balanced metric on the Iwasawa manifold has a positive γ_1 number. In section 7, we establish the existence of 1-st Gauduchon metric on Calabi's 3-dimensional non-Kähler manifold, by using Theorem 6 and proving that the balanced metric on the manifold has a negative γ_1 number. In the last section, we prove the existence of a 1-st Gauduchon metric on $S^5 \times S^1$. We also show the nonexistence of balanced metric and pluriclosed metric on $S^5 \times S^1$.

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2. Notation and preliminaries

Throughout this note, we use the following convention: We write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Let $(g^{i\bar{j}})$ be the transposed inverse of the matrix $(g_{i\bar{j}})$. For any two real 1-forms A and B on X, locally given by

$$A = \sum_{i=1}^{n} \big(A_i dz_i + A_{\bar{i}} d\bar{z}_i \big) \quad \text{and} \quad B = \sum_{i=1}^{n} \big(B_i dz_i + B_{\bar{i}} d\bar{z}_i \big),$$

we denote

$$\langle A, B \rangle_\omega = \frac{1}{2} \sum_{i,j=1}^{n} g^{i\bar{j}} \big(A_i B_{\bar{j}} + A_{\bar{j}} B_i \big).$$

We may omit the subscript ω in $\langle \cdot, \cdot \rangle_\omega$ when it is understood from the context. In particular, we have

$$\langle dh, dh \rangle = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial h}{\partial z_i} \frac{\partial h}{\partial \bar{z}_j} \equiv |\nabla h|^2, \qquad \text{for all } h \in C^1(X).$$

The Laplacian Δ associated with ω is given by

$$\Delta h = \frac{n\omega^{n-1} \wedge (\sqrt{-1}/2)\partial\bar{\partial} h}{\omega^n} = \sum_{i,j=1}^{n} g^{i\bar{j}} h_{i\bar{j}}, \qquad \text{for all } h \in C^2(X).$$

We use the continuity method to solve (1.10). Fix an integer $l \geq n+4$ and a real number $0 < \alpha < 1$. We denote by $C^{l,\alpha}(X)$ the usual Hölder space on X. Let

$$S(u) = \Delta u + \psi(|\nabla u|^2) + \langle B, du\rangle - \frac{\int_X (\Delta u + \psi(|\nabla u|^2) + \langle B, du\rangle)\,\omega^n}{\int_X \omega^n},$$

for each $u \in C^{l,\alpha}(X)$. Consider the following family of equations,

$$S(v_t) = tf, \qquad 0 \leq t \leq 1. \tag{2.1}$$

Let I be the subset of $[0,1]$ consisting of t for which the equation (2.1) has a solution $v_t \in C^{l,\alpha}(X)$ satisfying

$$\int_X v_t\,\omega^n = 0. \tag{2.2}$$

Obviously, the set I is nonempty since $0 \in I$. The openness of I will follow from our previous results [9, Section 3]. Indeed, let

$$\mathcal{E}_\omega^{l,\alpha} = \left\{ h \in C^{l,\alpha}(X); \int_X h\omega^n = 0 \right\}. \tag{2.3}$$

Notice that $S : \mathcal{E}_\omega^{l+2,\alpha} \to \mathcal{E}_\omega^{l,\alpha}$. The linearization of S is

$$L_\omega(h) = \frac{d}{dt} S(v+th)\bigg|_{t=0} = \Delta h + \langle \tilde{B}, dh\rangle - \frac{\int_X (\Delta h + \langle \tilde{B}, dh\rangle)\omega^n}{\int_X \omega^n},$$

where

$$\tilde{B} = B + 2\psi'(|\nabla v|^2)\,dv.$$

It follows from the proof of Lemma 13 in [9] that L_ω is a linear isomorphism from $\mathcal{E}^{l+2,\alpha}(X)$ to $\mathcal{E}^{l,\alpha}(X)$. Thus, by the implicit theorem we obtain the openness of I.

For the closedness of I we need the *a priori* estimate, which will be established in Section 3.

3. A Prior estimates

Let (X,ω) be an n-dimensional hermitian manifold, B a smooth 1-form on X, f a smooth function on X, c a constant, and $\psi \in C^\infty(\mathbb{R})$ satisfy (1.9). Consider the following semi-linear equation:

$$S(v) \equiv \Delta v + \psi(|\nabla v|^2) + \langle B, dv\rangle - c = f \qquad \text{on } X, \tag{3.1}$$

where $v \in C^3(X)$ satisfies the normalization condition

$$\int_X v\,\omega^n = 0. \tag{3.2}$$

We shall first derive a uniform gradient estimate:

Lemma 7. *Let $v \in C^3(X)$ be a solution of (3.1). We have*

$$\sup_X |\nabla v| \leq C,$$

where $C > 0$ is a constant depending only on B, f, ω, $\psi(0)$, μ and ν.

Throughout this section, we always denote by $C > 0$ a generic constant depending only on B, f, ω, $\psi(0)$, μ, and ν, unless otherwise indicated.

Proof. Since X is compact, we can assume that $|\nabla v|^2$ attains its maximum at some point $x_0 \in X$. Consider the following linear elliptic operator

$$L(h) = \Delta h + 2\psi'(|\nabla v|^2)\langle dh, dv\rangle_\omega = \Delta h + \psi'(|\nabla v|^2) g^{i\bar{j}}(h_i v_{\bar{j}} + h_{\bar{j}} v_i),$$

Here the summation convention is used, and we denote

$$h_i = \frac{\partial h}{\partial z^i}, \quad g^{i\bar{j}}_{,k} = \frac{\partial g^{i\bar{j}}}{\partial z^k}, \quad \cdots .$$

We compute that

$$\begin{aligned} L(|\nabla v|^2) &= \Delta(|\nabla v|^2) + \psi' g^{i\bar{j}}\big[(|\nabla v|^2)_i v_{\bar{j}} + v_i(|\nabla v|^2)_{\bar{j}}\big] \\ &= g^{i\bar{j}} g^{p\bar{q}}(v_{pi} v_{\bar{q}\bar{j}} + v_{p\bar{j}} v_{i\bar{q}}) + g^{p\bar{q}}\big[(\Delta v)_p v_{\bar{q}} + v_p(\Delta v)_{\bar{q}}\big] + g^{i\bar{j}} g^{p\bar{q}}_{,i\bar{j}} v_p v_{\bar{q}} \\ &\quad + g^{i\bar{j}} g^{p\bar{q}}_{,i}(v_{p\bar{j}} v_{\bar{q}} + v_p v_{\bar{q}\bar{j}}) + g^{i\bar{j}} g^{p\bar{q}}_{,\bar{j}}(v_{pi} v_{\bar{q}} + v_p v_{i\bar{q}}) \\ &\quad - g^{p\bar{q}}(g^{i\bar{j}}_{,p} v_{i\bar{j}} v_{\bar{q}} + g^{i\bar{j}}_{,\bar{q}} v_p v_{i\bar{j}}) + \psi' g^{i\bar{j}}\big[(|\nabla v|^2)_i v_{\bar{j}} + v_i(|\nabla v|^2)_{\bar{j}}\big]. \end{aligned}$$

Using equation (3.1) to the second term on the far right of above equalities and then using the Schwarz inequality, we find

$$L(|\nabla v|^2) \geq \frac{1}{2} g^{i\bar{j}} g^{p\bar{q}}(v_{pi} v_{\bar{q}\bar{j}} + v_{p\bar{j}} v_{i\bar{q}}) - C|\nabla v|^2 - C.$$

To see more clearly, let us take a normal coordinate system around x_0 such that

$$g_{i\bar{j}}(x_0) = \delta_{ij}, \qquad \text{for all } i, j = 1, \ldots, n.$$

It follows that

$$\begin{aligned} L(|\nabla v|^2) &\geq \frac{1}{2}\sum_{i,p=1}^{n} |v_{p\bar{i}}|^2 - C|\nabla v|^2 - C \\ &\geq \frac{1}{2}\sum_{i=1}^{n} |v_{i\bar{i}}|^2 - C|\nabla v|^2 - C \\ &\geq \frac{1}{2n}|\Delta v|^2 - C|\nabla v|^2 - C \qquad \big(\text{by Cauchy's inequality}\big) \\ &\geq \frac{1}{2n}\big|\psi(|\nabla v|^2) + \langle B, dv\rangle - f - c\big|^2 - C|\nabla v|^2 - C \qquad \big(\text{by (3.1)}\big) \\ &\geq \frac{1}{4n}\big|\psi(|\nabla v|^2)\big|^2 - C|\nabla v|^2 - C(1 + |c|^2). \end{aligned}$$

We can assume, without loss of generality, that $|\nabla v|^2(x_0)$ is sufficiently large so that

$$\psi(|\nabla v|^2) \geq \frac{\nu}{2}|\nabla v|^{2\mu} \qquad \text{at } x_0,$$

where $\mu > 1/2$ and $\nu > 0$ are constants, by (1.9). Now notice that

$$L(|\nabla v|^2) \leq 0 \quad \text{at } x_0,$$

because of

$$\Delta(|\nabla v|^2)(x_0) \leq 0, \qquad \text{and} \quad \nabla(|\nabla v|^2)(x_0) = 0.$$

Hence, we obtain that

$$\sup_X |\nabla v|^2 = |\nabla v|^2(x_0) \leq C(1 + |c|^2).$$

It remains to bound the constant c in terms of f and $\psi(0)$: Apply the usual maximum principle to (3.1) to obtain that

$$\psi(0) - \sup_X f \leq c \leq - \inf_X f + \psi(0). \tag{3.3}$$

This finishes the proof. □

Next, we establish the C^0 estimate: Noticing (3.2), there must exist some point $y_0 \in X$ such that $v(y_0) = 0$. Then, for any point $y \in X$, we take a geodesic curve γ connecting y_0 to y. We have by Lemma 7 that,

$$|v(y)| = |v(y) - v(y_0)| = \left| \int_0^1 \frac{d(v \circ \gamma)}{dt} dt \right| \leq \int_0^1 (|\nabla v| \circ \gamma)\, dt < C.$$

This settles the C^0 estimate of v.

We rewrite equation (3.1) as

$$\triangle v = -\psi(|\nabla v|^2) - \langle B, dv \rangle + f + c.$$

By $W^{2,p}$ theory of elliptic equations, we have for any $p > 1$,

$$\begin{aligned} \|v\|_{W^{2,p}} &\leqq C(\|v\|_{L^p} + \|f + c - \psi(|\nabla v|^2) - \langle B, dv \rangle\|_{L^p}) \\ &\leq C_1, \end{aligned}$$

where in the last inequality we have used the C^0 and C^1 estimates of v, and (3.3). Here and below, we denote by C_1 a generic constant depending on B, f, ω, μ, ν, and also p, and $\max\{|\psi(t)|; 0 \leq t \leq \max |\nabla v|^2 \leq C\}$.

Fix a sufficiently large p such that $\alpha \equiv 2n/p < 1$. It follows from the Sobolev embedding theorem that

$$\|v\|_{C^{1,\alpha}} \leq C_1.$$

This allows us to apply Schauder's theory to obtain that

$$\|v\|_{C^{2,\alpha}} \leq C_1.$$

Thus, by the bootstrap argument, we have

$$\|v\|_{C^{l,\alpha}} \leq C_1, \qquad \text{for any } k \geq 1. \tag{3.4}$$

This implies that the set I defined in Section 2 is closed. As a consequence, we have shown the existence part in Theorem 2.

4. Uniqueness and Corollary

Let us prove the uniqueness in Theorem 2. Suppose that there exist c, v and $\tilde{c}$, $\tilde{v}$ such that

$$\Delta v + \psi(|\nabla v|^2) + \langle B, dv\rangle = f + c,$$
$$\Delta \tilde{v} + \psi(|\nabla \tilde{v}|^2) + \langle B, d\tilde{v}\rangle = f + \tilde{c}.$$

Then,

$$c = \frac{\int_X (\Delta v + \psi(|\nabla v|^2) + \langle B, dv\rangle)\omega^n}{\int_X \omega^n}, \tag{4.1}$$

$$\tilde{c} = \frac{\int_X (\Delta \tilde{v} + \psi(|\nabla \tilde{v}|^2) + \langle B, d\tilde{v}\rangle)\omega^n}{\int_X \omega^n}. \tag{4.2}$$

Recall that we denote

$$S(u) = \Delta u + \psi(|\nabla u|^2) + \langle B, du\rangle - \frac{\int_X (\Delta u + \psi(|\nabla u|^2) + \langle B, du\rangle)\omega^n}{\int_X \omega^n},$$

for all $u \in C^2(X)$. It follows that

$$\begin{aligned} 0 = S(v) - S(\tilde{v}) &= \int_0^1 \left[\frac{d}{dt} S\big(tv + (1-t)\tilde{v}\big)\right] dt \\ &= \Delta w + \langle \tilde{B}, dw\rangle - c_w. \end{aligned} \tag{4.3}$$

Here $w = v - \tilde{v}$,

$$\tilde{B} = B + 2\int_0^1 \psi'(|t\nabla v + (1-t)\nabla \tilde{v}|^2)\big[t dv + (1-t) d\tilde{v}\big] dt,$$

and c_w is a constant given by

$$c_w = \frac{\int_X (\Delta w + \langle \tilde{B}, dw\rangle)\omega^n}{\int_X \omega^n}.$$

Applying the maximum principle to (4.3) yields

$$c_w = 0.$$

Then, by the strong maximum principle we conclude that w is equal to a constant. This shows that the solution of (1.10) is unique up to a constant. By (4.1) and (4.2) we have $c = \tilde{c}$. This completes the proof of Theorem 2.

Let us now prove Corollary 4. We define a smooth real 1-form on X

$$B_1 = \frac{\sqrt{-1}}{2}\frac{nk}{n-1}\frac{1}{n!} * \big(\partial(\omega^{n-1}) - \bar{\partial}(\omega^{n-1})\big) \tag{4.4}$$

and a smooth function

$$\varphi = \frac{n(\sqrt{-1}/2)\partial\bar{\partial}(\omega^k) \wedge \omega^{n-k-1}}{\omega^n}. \tag{4.5}$$

Then (1.11) is equivalent to

$$\Delta v + |\nabla v|^2 + \langle B_1, dv\rangle + \varphi = n\gamma_k.$$

Letting

$$\psi(t) = t \quad \text{and} \quad f = \frac{\int_X \varphi\omega^n}{\int_X \omega^n} - \varphi,$$

Corollary 4 then follows readily from Theorem 2.

For each $1 \le k \le n-1$, the constant γ_k is given by

$$\gamma_k = \frac{\int_X e^{-v}(\sqrt{-1}/2)\partial\bar\partial(e^v\omega^k) \wedge \omega^{n-k-1}}{\int_X \omega^n} \tag{4.6}$$

$$= \frac{\int_X (\Delta v + |\nabla v|^2 + \langle B_1, dv\rangle + \varphi)\omega^n}{n\int_X \omega^n}. \tag{4.7}$$

On the other hand, directly integrating (1.11) over X yields that

$$\gamma_k = \frac{\int_X (\sqrt{-1}/2)\partial\bar\partial(e^v\omega^k) \wedge \omega^{n-k-1}}{\int_X e^v\omega^n}. \tag{4.8}$$

This together with (4.6) imposes some constraint on the constant γ_k. For instance, when $k = n-1$, by (4.8) we know that

$$\gamma_{n-1} = 0.$$

Thus, in this case Corollary 4 recovers the classic result of Gauduchon [11]. When ω is Kähler, by (4.8) again we have

$$\gamma_k = 0 \qquad \text{for all } 1 \le k \le n-1.$$

Then, it follows from (4.7) that

$$\int_X |\nabla v|^2\omega^n = 0.$$

This tells us that the solution v of (1.11) has to be a constant.

5. Generalized Gauduchon metrics and γ_k

Let X be an n-dimensional complex manifold. We recall by Definition 1 that a hermitian metric ω on X is called k-th Gauduchon metric if

$$\partial\bar\partial(\omega^k) \wedge \omega^{n-k-1} = 0 \qquad \text{on } X.$$

Then, the $(n-1)$–th Gauduchon metric is the Gauduchon metric in the usual sense. By Corollary 4, each hermitian metric ω on X can be associated with a unique constant $\gamma_k(\omega)$, which is invariant under biholomorphisms. The induced function $\gamma_k = \gamma_k(\omega)$ can be used to characterize the k-th Gauduchon metric.

Proposition 8. *The hermitian manifold X admits a k-th Gauduchon metric if and only if that there exists a hermitian metric ω on X such that*

$$\gamma_k(\omega) = 0. \tag{5.1}$$

Proof. If there is some hermitian metric ω satisfying (5.1), then Corollary 4 implies that the conformal metric $e^{v/k}\omega$ is a k-th Gauduchon metric on X. Conversely, if ω is a k-th Gauduchon metric, then the uniqueness of Corollary 4 implies that $\gamma_k(\omega) = 0$ and that v is a constant. □

Let $\mathfrak{M}$ be the set of all hermitian metrics on X. We shall prove that γ_k is a smooth function on $\mathfrak{M}$. Here $\mathfrak{M}$ is viewed as an open subset in $C^{l+2,\alpha}(\Lambda^{1,1}_{\mathbb{R}}(X))$, for a nonnegative integer l and a real number $0<\alpha<1$. We denote by $C^{l,\alpha}(\Lambda^{m,m}_{\mathbb{R}}(X))$ the Hölder space of real (m,m)–forms on X, in which l and m are nonnegative integers, and $0<\alpha<1$ is a real number. In particular, $C^{l,\alpha}(\Lambda^{0,0}_{\mathbb{R}}(X)) = C^{l,\alpha}(X)$.

Proposition 9. *The function $\gamma_k = \gamma_k(\omega)$ is a smooth function on $\mathfrak{M}$, where $\mathfrak{M}$ is viewed as an open subset in $C^{l+2,\alpha}(\Lambda^{1,1}_{\mathbb{R}}(X))$.*

Proof. It follows from Corollary 4 that, for each $\omega \in \mathfrak{M}$, there exists a unique constant γ_k and a function v such that

$$e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k)\wedge\omega^{n-k-1} - \gamma_k\omega^n = 0. \tag{5.2}$$

Then,

$$\gamma_k = \frac{\int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k)\wedge\omega^{n-k-1}}{\int_X \omega^n}$$

depends smoothly on v and ω. Thus, to show the result, it suffices to show that the solution v depends smoothly on ω. We shall use the implicit function theorem.

For each $\omega \in \mathfrak{M}$, the space $\mathcal{E}^{l,\alpha}_{\omega}$ is defined by (2.3). Fix $\omega_0 \in \mathfrak{M}$, for which we abbreviate $\mathcal{E}^{l,\alpha}_0 = \mathcal{E}^{l,\alpha}_{\omega_0}$. We have two obvious linear isomorphisms from $\mathcal{E}^{l,\alpha}_{\omega}$ to $\mathcal{E}^{l,\alpha}_0$, given respectively by

$$h \longmapsto h - \frac{\int_X h\omega_0^n}{\int_X \omega_0^n}, \qquad \text{for all } h \in \mathcal{E}^{l,\alpha}_{\omega}, \tag{5.3}$$

and

$$h \longmapsto h\cdot\frac{\omega^n}{\omega_0^n} \qquad \text{for all } h \in \mathcal{E}^{l,\alpha}_{\omega}. \tag{5.4}$$

Define a map $F : \mathfrak{M}\times\mathcal{E}^{l+2,\alpha}_0 \to \mathcal{E}^{l,\alpha}_0$ by

$$\begin{aligned} F(\omega,v) = {} & \frac{ne^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k)\wedge\omega^{n-k-1}}{\omega_0^n} \\ & - \frac{n\int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v\omega^k)\wedge\omega^{n-k-1}}{\int_X\omega^n}\cdot\frac{\omega^n}{\omega_0^n}. \end{aligned}$$

Obviously, F is a smooth map. Note that any $(\omega,v)\in\mathfrak{M}\times\mathcal{E}^{l+2,\alpha}_0$ satisfies (5.2) if and only if

$$F(\omega,v)=0.$$

The Fréchet derivative of F with respect to the variable v is

$$D_vF(\omega,v)(h) = L_{\omega}(h)\frac{\omega^n}{\omega_0^n}.$$

Here

$$L_{\omega}(h) = \Delta h + \langle B_1 + 2dv, dh\rangle_{\omega} - \frac{\int_X(\Delta h + \langle B_1+2dv,dh\rangle_{\omega})\omega^n}{\int_X\omega^n},$$

in which the Laplacian Δ is with respect to ω, and B_1 is the smooth real 1-form given by (4.4). By the proof of Lemma 13 in [9] and the isomorphism (5.3), the operator $L_\omega : \mathcal{E}_0^{l+2,\alpha} \to \mathcal{E}_\omega^{l,\alpha}$ is a linear isomorphism. This combining isomorphism (5.4) imply that $D_v F(\omega, v) : \mathcal{E}_0^{l+2,\alpha} \to \mathcal{E}_0^{l,\alpha}$ is a linear isomorphism. The result then follows by the Implicit Function Theorem. □

A direct corollary of Proposition 9 is as below.

Corollary 10. *For $1 \le k \le n-2$, if there exists two hermitian metric ω_1, ω_2 on X such that*

$$\gamma_k(\omega_1) > 0 \quad \textit{and} \quad \gamma_k(\omega_2) < 0,$$

then there exists a metric ω on X satisfying $\gamma_k(\omega) = 0$, i.e., ω is a k-th Gauduchon metric.

Proof. Let

$$\omega_t = t\omega_1 + (1-t)\omega_2, \qquad \text{for all } 0 \le t \le 1.$$

Then ω_t is a hermitian metric for each t. The result follows immediately by applying the Mean Value Theorem to the function $\phi(t) = \gamma_k(\omega_t)$. □

Proposition 11. *For any function $\rho \in C^2(M)$, we have*

$$e^{-\max_X \rho}\gamma_k(\omega) \le \gamma_k(e^\rho \omega) \le e^{-\min_X \rho}\gamma_k(\omega). \tag{5.5}$$

In particular, the sign of the function γ_k is a conformal invariant for hermitian metrics.

Proof. Let $\tilde\omega = e^\rho\omega$. Then, there exists a function $\tilde v$ and a number $\tilde\gamma_k = \gamma_k(\tilde\omega)$ satisfying

$$(\sqrt{-1}/2)\partial\bar\partial(e^{\tilde v}\tilde\omega^k) \wedge \tilde\omega^{n-k-1} = \tilde\gamma_k e^{\tilde v}\tilde\omega^n,$$

that is,

$$(\sqrt{-1}/2)\partial\bar\partial(e^{\tilde v + k\rho}\omega^k) \wedge \omega^{n-k-1} = \tilde\gamma_k e^{\tilde v + k\rho} e^\rho \omega^n. \tag{5.6}$$

We can rewrite (5.6) as

$$\Delta(\tilde v + k\rho) + |\nabla(\tilde v + k\rho)|^2 + \langle B_1, d(\tilde v + k\rho)\rangle + \varphi = ne^\rho\tilde\gamma, \tag{5.7}$$

where the operators Δ and ∇ are with respect to ω, and B_1 and φ are given by (4.4) and (4.5), respectively. Subtracting (5.7) by

$$\Delta v + |\nabla v|^2 + \langle B_1, dv\rangle + \varphi = n\gamma_k(\omega)$$

and then applying the maximum principle yields (5.5). □

Proposition 12. *For a hermitian metric ω, the number $\gamma_k(\omega) > 0$ $(= 0$, or $< 0)$ if and only if there exists a metric $\tilde\omega$ in the conformal class of ω such that*

$$(\sqrt{-1}/2)\partial\bar\partial\tilde\omega^k \wedge \tilde\omega^{n-k-1} > 0 \,(= 0, \text{ or } < 0) \quad \text{on } X. \tag{5.8}$$

Proof. Suppose that $\gamma_k(\omega) > 0$ $(= 0$, or $< 0)$. Let $\tilde\omega = e^{v/k}\omega$, where v is the smooth function associated with ω so that (1.11) holds. Then,

$$(\sqrt{-1}/2)\partial\bar\partial\tilde\omega^k \wedge \tilde\omega^{n-k-1} = \gamma_k(\omega)\omega^n e^{(n-k)v} > 0\,(= 0, \text{ or } < 0).$$

Conversely, if there is a metric $\tilde\omega$ in the conformal class of ω such that (5.8) holds, then we claim that $\gamma_k(\tilde\omega) > 0$ $(= 0$, or $< 0)$. Indeed, by Corollary 4 there exists a smooth function $\tilde v$ such that

$$(\sqrt{-1}/2)\partial\bar\partial(e^{\tilde v}\tilde\omega^k) \wedge \tilde\omega^{n-k-1} = \gamma_k(\tilde\omega)e^{\tilde v}\tilde\omega.$$

This is equivalent to the following equation

$$\Delta\tilde v + |\nabla\tilde v|^2 + \langle\tilde B_1, d\tilde v\rangle + \tilde\varphi = n\gamma_k(\tilde\omega), \tag{5.9}$$

where the operators Δ and ∇ are with respect to $\tilde\omega$, and $\tilde B_1$ and $\tilde\varphi$ are given by (4.4) and (4.5), respectively, with $\tilde\omega$ replacing ω. By (5.8) we have $\tilde\varphi > 0$ $(= 0$, or $< 0)$. The claim then follows immediately by applying the maximum principle to (5.9). By Proposition 11, we finish the proof. □

Moreover, for the case of $\gamma_k > 0$, we have the following criteria on the integration, which is often easier to verify.

Lemma 13. *Suppose that n, the complex dimension of X, is an odd number. Let $k = (n-1)/2$. Then, there is some metric ω satisfying $\gamma_k(\omega) > 0$ if and only if there is some semi-metric $\mathring\omega$ (i.e., semi-positive real $(1,1)$-form on X) satisfying*

$$\frac{\sqrt{-1}}{2}\int_X \partial\bar\partial\mathring\omega^k \wedge \mathring\omega^{n-k-1} > 0$$

Proof. By Proposition 12, the necessary part is obvious. For the sufficient part, let $\hat\omega$ be any hermitian metric. Let

$$\omega_t = \mathring\omega + t\hat\omega$$

for $t \in (0,1)$. Then we have

$$\begin{aligned}
&\int_X e^{-v}(\sqrt{-1}/2)\partial\bar\partial(e^v\omega_t^k) \wedge \omega_t^{n-k-1} \\
&= \frac{\sqrt{-1}}{2}\int_X \left(\partial\bar\partial\omega_t^k \wedge \omega_t^{n-k-1} + \partial v \wedge \bar\partial v \wedge \omega_t^{n-1}\right) \\
&\quad + \frac{\sqrt{-1}}{2}\int_X \left[\partial\bar\partial v \wedge \omega_t^{n-1} + \frac{k}{n-1}\left(\partial\omega_t^{n-1} \wedge \bar\partial v + \partial v \wedge \bar\partial\omega_t^{n-1}\right)\right] \\
&= \frac{\sqrt{-1}}{2}\int_X \left(\partial\bar\partial\omega_t^k \wedge \omega_t^{n-k-1} + \partial v \wedge \bar\partial v \wedge \omega_t^{n-1}\right) \\
&\quad + \frac{\sqrt{-1}}{2}\Big(1 - \frac{2k}{n-1}\Big)\int_X v\partial\bar\partial\omega_t^{n-1}.
\end{aligned} \tag{5.10}$$

Since $k=(n-1)/2$, the second integral on the right of (5.10) vanishes. It follows that

$$\begin{aligned}\int_X e^{-v}(\sqrt{-1}/2)\partial\bar\partial(e^v\omega_t^k)\wedge\omega_t^{n-k-1} &\geq \frac{\sqrt{-1}}{2}\int_X \partial\bar\partial\omega_t^k\wedge\omega_t^k\\ =\frac{\sqrt{-1}}{2}\int_X\partial\bar\partial\mathring{\omega}^k\wedge\mathring{\omega}^k &+ t\frac{\sqrt{-1}}{2}\int_X(\partial\bar\partial\mathring{\omega}^k\wedge\Psi_t+\partial\bar\partial\Psi_t\wedge\mathring{\omega}^k)\\ &+t^2\frac{\sqrt{-1}}{2}\int_X\partial\bar\partial\Psi_t\wedge\Psi_t>0, \qquad \text{for sufficiently small } t,\end{aligned}$$

where $\Psi_t=\hat\omega\wedge(\mathring{\omega}^{k-1}+\mathring{\omega}^{k-2}\wedge\omega_t+\cdots+\mathring{\omega}\wedge\omega_t^{k-2}+\omega_t^{k-1})$. This implies that $\gamma_k(\omega_t)>0$ for the sufficiently small t. □

A similar argument works for the (classic) Gauduchon metrics, for any dimension n, and for all $1\leq k\leq n-2$.

Lemma 14. *Let X be an n-dimensional hermitian manifold, k an integer such that $1\leq k\leq n-2$. Then, a hermitian metric ω on X satisfies $\gamma_k(\omega)>0$ if the Gauduchon metric $\tilde\omega$ in the conformal class of ω satisfies*

$$\frac{\sqrt{-1}}{2}\int_X\partial\bar\partial\tilde\omega^k\wedge\tilde\omega^{n-k-1}>0. \tag{5.11}$$

Proof. By Proposition 11, we can assume that $\omega=\tilde\omega$, without loss of generality. By (5.10) with ω replacing ω_t, and applying $\partial\bar\partial\omega^{n-1}=0$ yields

$$\int_X e^{-v}(\sqrt{-1}/2)\partial\bar\partial(e^v\omega^k)\wedge\omega^{n-1-k}\geq\frac{\sqrt{-1}}{2}\int_X\partial\bar\partial\omega^k\wedge\omega^{n-1-k}>0.$$

□

Corollary 15. *Let (X,ω) be an n-dimensional balanced manifold. Then, for each $1\leq k\leq n-2$, we have $\gamma_k(\omega)>0$ if*

$$\frac{\sqrt{-1}}{2}\int_X\partial\bar\partial\omega^k\wedge\omega^{n-1-k}>0.$$

6. Constructions on hermitian three–manifolds

We shall apply previous results to construct a hermitian metric with $\gamma_1>0$ on a complex three dimensional manifold. Theorem 6 will follow from Proposition 12, together with the following theorem.

Theorem 16. *There always exists a hermitian metric ω on a complex three dimensional manifold X such that*

$$(\sqrt{-1}/2)\partial\bar\partial\omega\wedge\omega>0.$$

Proof. By Lemma 13 and Proposition 12, it suffices to construct a semi-metric $\mathring{\omega}$ such that

$$\frac{\sqrt{-1}}{2}\int_X\partial\bar\partial\mathring{\omega}\wedge\mathring{\omega}>0.$$

Fix a point $q \in X$ and a coordinate patch $U \ni q$. Let (z_1, z_2, z_3) be coordinates on U centered at q. Here $z_j = x_j + \sqrt{-1}y_j$ for $1 \leq j \leq 3$. We can assume $N = B \times B \times R \subset U$, where B is the unit ball in $\mathbb{C}$, and

$$R = \{z_3 \in \mathbb{C} \mid |x_3| \leq 1, |y_3| \leq 1\}.$$

Take a nonnegative cut-off function $\eta \in C_0^\infty(B)$ and two nonnegative functions $f, g \in C_0^\infty([-1, 1])$ to be determined later. On N, define

$$\phi = \eta(z_1)\eta(z_2)f(x_3)f(y_3), \quad \psi = \eta(z_1)\eta(z_2)g(x_3)g(y_3),$$

and then define

$$\mathring{\omega} = \frac{\sqrt{-1}}{2}\big[\phi(z)dz_1 \wedge d\bar{z}_1 + \psi(z)dz_2 \wedge d\bar{z}_2\big]. \tag{6.1}$$

Obviously, $\mathring{\omega}$ is semi-positive and with compact support in N. So it can be viewed as a semi-metric on X. Clearly,

$$\frac{\sqrt{-1}}{2}\partial\bar{\partial}\mathring{\omega} \wedge \mathring{\omega} = \Big(\phi\frac{\partial^2\psi}{\partial z_3\partial\bar{z}_3} + \psi\frac{\partial^2\phi}{\partial z_3\partial\bar{z}_3}\Big)dV, \tag{6.2}$$

where

$$dV = \Big(\frac{\sqrt{-1}}{2}\Big)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3. \tag{6.3}$$

Since

$$\frac{\partial}{\partial z_3} = \frac{1}{2}\Big(\frac{\partial}{\partial x_3} - \sqrt{-1}\frac{\partial}{\partial y_3}\Big), \quad \frac{\partial}{\partial \bar{z}_3} = \frac{1}{2}\Big(\frac{\partial}{\partial x_3} + \sqrt{-1}\frac{\partial}{\partial y_3}\Big),$$

we have

$$\begin{aligned}
&\phi\frac{\partial^2\psi}{\partial z_3\partial\bar{z}_3} + \psi\frac{\partial^2\phi}{\partial z_3\partial\bar{z}_3}\\
&= \frac{\phi}{4}\left(\frac{\partial^2\psi}{\partial x_3\partial x_3} + \frac{\partial^2\psi}{\partial y_3\partial y_3}\right) + \frac{\psi}{4}\left(\frac{\partial^2\phi}{\partial x_3\partial x_3} + \frac{\partial^2\phi}{\partial y_3\partial y_3}\right)\\
&= \frac{1}{4}\eta^2(z_1)\eta^2(z_2)f(y_3)g(y_3)\big[f(x_3)g''(x_3) + g(x_3)f''(x_3)\big]\\
&\quad + \frac{1}{4}\eta^2(z_1)\eta^2(z_2)f(x_3)g(x_3)\big[f(y_3)g''(y_3) + g(y_3)f''(y_3)\big].
\end{aligned}$$

We choose η so that

$$\int_B \eta^2(z)\frac{\sqrt{-1}}{2}dz \wedge d\bar{z} = 1.$$

Then it follows that

$$\begin{aligned}
\frac{\sqrt{-1}}{2}\int_X \partial\bar{\partial}\mathring{\omega} \wedge \mathring{\omega} &= \frac{1}{2}\int_{-1}^1 f(t)g(t)dt \int_{-1}^1 \big[f(t)g''(t) + f''(t)g(t)\big]dt\\
&= \int_{-1}^1 f(t)g(t)dt \int_{-1}^1 \big[-f'(t)g'(t)\big]dt.
\end{aligned}$$

The result follows immediately from the proposition below. $\square$

Proposition 17. *There exist nonnegative functions $f, g \in C_0^\infty([-1,1])$ such that*

$$-\int_{-1}^{1} f'(t)g'(t)dt > 0.$$

Proof. For any two real numbers $a < b$, we denote

$$\chi_{a,b}(t) = \begin{cases} \exp\left(\dfrac{1}{t-b} - \dfrac{1}{t-a}\right), & \text{if } a < t < b, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have that $\chi_{a,b} \in C_0^\infty(\mathbb{R})$, that $\chi'_{a,b}(t) > 0$ for $a < t < (a+b)/2$, that $\chi'_{a,b}(t) < 0$ for $(a+b)/2 < t < b$, and that $\chi'_{a,b}(t) = 0$ when $t = (a+b)/2$. Letting

$$f(t) = \chi_{-1/3,1/3}(t), \qquad \text{and} \qquad g(t) = \chi_{0,2/3}(t)$$

yields that $-f'(t)g'(t) > 0$ for $0 < t < 1/3$ and otherwise $f'(t)g'(t) = 0$. This in particular implies the result. $\square$

Let us now consider some examples. We can directly construct a hermitian metric ω with $\gamma_1(\omega) > 0$ on T^3, the 3-dimensional complex torus.

Proposition 18. *On the complex T^3, there is a metric ω satisfying*

$$\sqrt{-1}/2\partial\bar{\partial}\omega \wedge \omega > 0.$$

Proof. Let (z_1, z_2, z_3) be the coordinates of T^3 induced from $\mathbb{C}^3$. Let

$$\omega = \frac{\sqrt{-1}}{2}\left[\xi(x_3)dz_1 \wedge d\bar{z}_1 + \eta(x_3)dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3\right],$$

where ξ and η are two positive smooth functions on T^3 only depending on x_3, which will be determined later. Then

$$(\sqrt{-1}/2)\partial\bar{\partial}\omega \wedge \omega = \left(\eta\frac{\partial^2\xi}{\partial z_3\partial\bar{z}_3} + \xi\frac{\partial^2\eta}{\partial z_3\partial\bar{z}_3}\right)dV > 0$$

if and only if

$$\eta\frac{\partial^2\xi}{\partial z_3\partial\bar{z}_3} + \xi\frac{\partial^2\eta}{\partial z_3\partial\bar{z}_3} = \frac{1}{4}\eta\frac{\partial^2\xi}{\partial x_3^2} + \frac{1}{4}\xi\frac{\partial^2\eta}{\partial x_3^2} > 0.$$

Here dV is defined by (6.3). So we need to look for two smooth, positive, 2π-periodic functions η and ξ such that

$$\frac{\eta''(t)}{\eta(t)} + \frac{\xi''(t)}{\xi(t)} > 0.$$

We define

$$\xi(t) = 1 + \kappa\sin t, \qquad \text{for some } 0 < \kappa < 1. \tag{6.4}$$

We observe that

$$\int_0^{2\pi}\frac{\xi''}{\xi}dt = -\int_0^{2\pi}\frac{\kappa\sin t}{1+\kappa\sin t}dt = -2\pi + \int_0^{2\pi}\frac{dt}{1+\kappa\sin t}.$$

By Proposition 8 in [9], the value of above integral tends to $+\infty$ monotonically, as $\kappa \to 1^-$. Hence, for a constant $C > 0$, there is a unique real number κ, such that the function ξ given by (6.4) satisfies

$$\int_0^{2\pi} \frac{\xi''}{\xi} dt = \int_0^{2\pi} C dt.$$

It implies that equation

$$\zeta'' + \frac{\xi''}{\xi} = C$$

has a smooth 2π-periodic solution ζ on $\mathbb{R}$. Let $\eta = e^{\zeta}$. Thus,

$$\frac{\eta''(t)}{\eta(t)} + \frac{\xi''(t)}{\xi(t)} \quad = \quad (\zeta')^2 + \zeta'' + \frac{\xi''}{\xi} \quad \geq \quad C \quad > \quad 0.$$

□

As another example, we show that the natural balanced metric on the Iwasawa manifold has positive γ_1. Recall (for example, [16, p. 444] and [20, p. 115]) that the Iwasawa manifold is defined to be the quotient space G/Γ, where

$$G = \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} ; z_1, z_2, z_3 \in \mathbb{C} \right\},$$

Γ is the discrete subgroup of G consisting of matrices where z_1, z_2, z_3 are Gaussian integers, i.e., $z_i \in \{a + b\sqrt{-1} \mid a, b \in \mathbb{Z}\}$ for $1 \leq i \leq 3$, and Γ acts on G by left multiplications. Clearly, the global holomorphic 1-forms

$$\varphi_1 = dz_1, \qquad \varphi_2 = dz_2, \qquad \varphi_3 = dz_3 - z_1 dz_2$$

on G are invariant under the action of Γ, hence descend down to G/Γ. Observe that G/Γ does not admit any Kähler metric, because $d\varphi_3 = \varphi_2 \wedge \varphi_1 \neq 0$. Let

$$\omega = (\sqrt{-1}/2)(\varphi_1 \wedge \bar{\varphi}_1 + \varphi_2 \wedge \bar{\varphi}_2 + \varphi_3 \wedge \bar{\varphi}_3).$$

Then, ω is a balanced hermitian metric on G/Γ, for $d\omega^2 = 0$. Furthermore, we have

$$(\sqrt{-1}/2)\partial\bar{\partial}\omega \wedge \omega = (\sqrt{-1}/2)^3 \varphi_1 \wedge \bar{\varphi}_1 \wedge \varphi_2 \wedge \bar{\varphi}_2 \wedge \varphi_3 \wedge \bar{\varphi}_3 > 0$$

on G/Γ; hence, by Proposition 12, we conclude that $\gamma_1(\omega) > 0$.

7. The first Gauduchon metric on Calabi's manifolds

In this section, we shall establish the existence of the 1-st Gauduchon metric on the non-Kähler manifold introduced by Calabi [5]. In view of Theorem 6 and Corollary 10, we need to find a hermitian metric with negative γ_1 value.

We first recall Calabi's construction of non-Kähler complex three dimensional manifolds. Let $\mathbb{O} \cong \mathbb{R}^8$ denotes the Cayley numbers. We fix a basis $\{I_1, \cdots, I_7\}$ such that

(1) $I_i \cdot I_j = \delta_{ij}$ with respect to the inner product.
(2) The table of the multiplication of the cross product $I_j \times I_k$ is the following

$$\begin{array}{c|ccccccc} \times & I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 \\ \hline I_1 & 0 & I_3 & -I_2 & I_5 & -I_4 & I_7 & -I_6 \\ I_2 & -I_3 & 0 & I_1 & I_6 & -I_7 & -I_4 & I_5 \\ I_3 & I_2 & -I_1 & 0 & -I_7 & -I_6 & I_5 & I_4 \\ I_4 & -I_5 & -I_6 & I_7 & 0 & I_1 & I_2 & -I_3 \\ I_5 & I_4 & I_7 & I_6 & -I_1 & 0 & -I_3 & -I_2 \\ I_6 & -I_7 & I_4 & -I_5 & -I_2 & I_3 & 0 & I_1 \\ I_7 & I_6 & -I_5 & -I_4 & I_3 & I_2 & -I_1 & 0 \end{array} \tag{7.1}$$

Via this basis, we have the isomorphism $\mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$.

Calabi considered a smooth oriented hypersurface $X^6 \hookrightarrow \mathbb{R}^7$. Fix a unit normal vector field N of X. There is a natural almost complex structure $J : TX \to TX$ induced by Cayley multiplication as follows. For any $x \in X$ and any $V \in T_xX$, define $J : T_xX \to T_xX$ as

$$J(V) = N \times V.$$

Calabi proved that J is integrable if and only if J anticommutes with the second fundamental form of X.

Calabi constructed compact complex manifolds as follows. Let Σ be a compact Riemann surface which admits 3 holomorphic differentials ϕ_1, ϕ_2, ϕ_3 with the following properties:

(1) linear independent;
(2) $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$;
(3) $\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2 + \phi_3 \wedge \bar{\phi}_3 > 0$.

Lifting ϕ_1, ϕ_2, ϕ_3 to the universal covering $\tilde{\Sigma} \to \Sigma$ and setting

$$x^j(p) = \mathrm{Re} \int_{p'}^{p} \phi_j, \quad j = 1, 2, 3$$

for a fixed point $p' \in \Sigma$, we obtain a conformal minimal immersion

$$\psi = (x^1, x^2, x^3) : \tilde{\Sigma} \to \mathbb{R}^3.$$

This mapping is regular, since the differentials ϕ_j satisfy (3); by the weierstrass representation, property (2) is equivalent to the statement that ψ is minimal; finally, because of property (1), it follows that $\tilde{\Sigma}$ is not mapped into a plane.

Calabi then considered the hypersurface of the type

$$(\psi, id) : \tilde{\Sigma} \times \mathbb{R}^4 \to \mathbb{R}^3 \times \mathbb{R}^4 = \mathrm{Im}(\mathbb{O}),$$

where $\mathbb{R}^3 = \mathrm{span}_{\mathbb{R}}\{I_1, I_2, I_3\}$ and $\mathbb{R}^4 = \mathrm{span}_{\mathbb{R}}\{I_4, I_5, I_6, I_7\}$. Since $\psi : \tilde{\Sigma} \to \mathbb{R}^3$ is minimal, $\tilde{\Sigma} \times \mathbb{R}^4$ is the complex manifold. If $g : \tilde{\Sigma} \to \tilde{\Sigma}$ denotes a covering transformation, then $\psi(gp) = \psi(p) + t_g$ for some vector $t_g \in \mathbb{R}^3$. It follows that the complex structure on $\tilde{\Sigma} \times \mathbb{R}^4$ is invariant by the covering group of Σ and so descends to $\Sigma \times \mathbb{R}^4$. On the other hand, for $\mathbb{R}^4$, we can further divide by a lattice Λ of translation of $\mathbb{R}^4$, and thereby produce a compact complex manifold $X_\Lambda = \Sigma \times T^4$. We can view X_Λ as a family of complex tori, parameterized by the Riemann surface.

Calabi showed that such complex manifolds X_Λ are non-Kähler. However, there exists a balanced metric on these manifolds [14, 19]. Let us consider the *nature* metric.

Define a 2-form on X_Λ as

$$\omega_0(V, W) = N \cdot (V \times W)$$

for any $V, W \in T_x X_\Lambda$ at any $x \in X_\Lambda$. Then clearly we have

$$\omega_0(V, W) = -\omega_0(W, V);$$

and using the formula

$$N \cdot (V \times W) = (N \times V) \cdot W,$$

we also have

$$\begin{aligned} &\omega_0(JV, JW) = \omega_0(V, W); \\ &\omega_0(V, JV) = (N \times V) \cdot (N \times V) > 0, \quad \text{if} \quad V \neq 0. \end{aligned}$$

So ω_0 is the positive $(1,1)$-form on X_Λ and therefore defines a hermitian metric.

Next we check that ω_0 is a balanced metric. The unit normal vector field of X in $\mathbb{R}^7$ can be written as

$$N = \sum_{j=1}^{3} a_j I_j, \qquad \sum_{j=1}^{3} a_j^2 = 1, \tag{7.2}$$

where a_j for $j = 1, 2, 3$ are functions on Σ. Let (x_4, x_5, x_6, x_7) be the coordinates of $\mathbb{R}^4$. Then we can write the hermitian metric ω_0 as

$$\omega_0 = \omega_\Sigma + \varphi_0,$$

where ω_Σ is a Kähler metric on Σ and

$$\begin{aligned} \varphi_0 = &a_1 dx_4 \wedge dx_5 + a_2 dx_4 \wedge dx_6 - a_3 dx_4 \wedge dx_7 \\ &- a_3 dx_5 \wedge dx_6 - a_2 dx_5 \wedge dx_7 + a_1 dx_6 \wedge dx_7. \end{aligned}$$

By direct check, we have

$$\varphi_0^2 = 2 dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7.$$

Therefore,

$$d(\omega_0^2) = d(2\omega_\Sigma \wedge \varphi_0 + \varphi_0^2) = 2d\omega_\Sigma \wedge \varphi_0 + 2\omega_\Sigma \wedge d\varphi_0 = 0,$$

since ω_Σ is a Kähler metric and all functions a_j are defined on Σ.

At last we prove that there exists a 1-Gauduchon metric on X_Λ. By direct computation, we have

$$\partial\bar{\partial}\omega_0 \wedge \omega_0 = \partial\bar{\partial}\varphi_0 \wedge \varphi_0 = 2\sum_{j=1}^{3} a_j \partial\bar{\partial} a_j \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7.$$

Condition (7.2) implies

$$\sum_{j=1}^{3} a_j \partial\bar{\partial} a_j = -\sum_{j=1}^{3} \partial a_j \wedge \bar{\partial} a_j,$$

Combining the above two equalities yields

$$\begin{aligned}\sqrt{-1}\partial\bar\partial\omega_0\wedge\omega_0 &= -2\sqrt{-1}\sum_{j=1}^{3}\partial a_j\wedge\bar\partial a_j\wedge dx_4\wedge dx_5\wedge dx_6\wedge dx_7\\ &= -4\sum_{j=1}^{3}|\partial a_j|^2\omega_0^3,\end{aligned}$$

and therefore,

$$\sqrt{-1}\int_{X_\Lambda}\partial\bar\partial(e^v\omega_0)\wedge\omega_0=\sqrt{-1}\int_{X_\Lambda}e^v\omega_0\wedge\partial\bar\partial\omega_0<0.$$

Hence, we have $\gamma_1(\omega_0)<0$, by Corollary 4; so $-1\in\Xi_1(X_\Lambda)$.

Proposition 19. $\Xi_1(X_\Lambda)=\{-1,0,1\}$.

Proof. We have proven $-1\in\Xi_1(X_\Lambda)$ and According to Theorem 6 we also have $1\in\Xi_1(X_\Lambda)$. Then by Corollary 10, $0\in\Xi_1(X_\Lambda)$. □

Corollary 20. *There exists a 1-Gauduchon metric on X_Λ.*

8. The first Gauduchon metric on $S^5\times S^1$

Let $S^5\to\mathbb{P}^2$ be the hopf fibration of the complex projective plane $\mathbb{P}^2$. S^5 can be viewed as the circle bundle over $\mathbb{P}^2$ twisted by $\frac{\omega_{FS}}{2\pi}\in H^2(\mathbb{P}^2,\mathbb{Z})$. Here ω_{FS} is the Fubini-Study metric on $\mathbb{P}^2$. We let $\pi: S^5\times S^1\to\mathbb{P}^2$ be the natural projection. Then using a canonical way (c.f. [10, 12]), we can define a complex structure on $S^5\times S^1$ such that π is a holomorphic map. We can define a natural hermitian metric on $S^5\times S^1$ as follows:

$$\omega_0=\pi^*\omega_{FS}+\frac{\sqrt{-1}}{2}\theta\wedge\bar\theta, \tag{8.1}$$

where $\theta=\theta_1+\sqrt{-1}\theta_2$ is a $(1,0)$-form on $S^5\times S^1$ such that $d\theta_1=\pi^*\omega_{FS}$ and $d\theta_2=0$. So $\bar\partial\theta=\pi^*\omega_{FS}$ and $\partial\theta=0$ which imply

$$\frac{\sqrt{-1}}{2}\partial\bar\partial\omega_0=-\frac{1}{4}\pi^*\omega_{FS}^2. \tag{8.2}$$

Thus

$$\frac{\sqrt{-1}}{2}\partial\bar\partial\omega_0\wedge\omega_0=\Big(\frac{\sqrt{-1}}{2}\Big)^3\pi^*\omega_{FS}^2\wedge\theta\wedge\bar\theta=-\frac{\omega_0^3}{3!} \tag{8.3}$$

and therefore

$$\sqrt{-1}\int_{S^5\times S^1}\partial\bar\partial(e^v\omega_0)\wedge\omega_0=\sqrt{-1}\int_{S^5\times S^1}e^v\omega_0\wedge\partial\bar\partial\omega_0<0.$$

Hence, we have $\gamma_1(\omega_0)<0$, by Corollary 4; so $-1\in\Xi_1(S^5\times S^1)$. Then by Corollary 10, $0\in\Xi_1(S^5\times S^1)$. That is we have

Proposition 21. *There exists a 1-Gauduchou metric on $S^5\times S^1$.*

Using above natural metric ω_0 on $S^5\times S^1$, we can also prove

Proposition 22. *There does not exist any pluri-closed metric on $S^5 \times S^1$.*

Proof. If there would exist a pluri-closed metric ω on $S^5 \times S^1$, then

$$0 = \int_{S^5\times S^1} \frac{\sqrt{-1}}{2}\partial\bar{\partial}\omega \wedge \omega_0 = -\frac{1}{4}\int_{S^5\times S^1} \omega \wedge \pi^*\omega_{FS}^2 < 0 \tag{8.4}$$

since $\omega \wedge \pi^*\omega_{FS}^2$ is the strictly positive definite $(3,3)$-form on $S^5 \times S^1$. That is a contradiction. $\square$

We also know that there does not exist any balanced metric on $S^5 \times S^1$. The proof is standard and is given here. There is an obstruction to the existence of a balanced metric on a compact complex manifold. Namely, on a compact complex manifold with a balanced metric no compact complex submanifold of codimension 1 can be homologous to 0 [19]. Now for $\pi : S^5 \times S^1 \to \mathbb{P}^2$, since π is a holomorphic, $\pi^{-1}(\mathbb{P}^2)$ is a complex hypersurface in $S^5 \times S^1$. Certainly $\pi^{-1}(\mathbb{P}^2)$ is homologous to zero in $S^5 \times S^1$ since $H^4(S^5 \times S^1, \mathbb{R}) = 0$. Therefor there exist no balanced metric on $S^5 \times S^1$.

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